Commutative n-ary superalgebras with an invariant skew-symmetric form[1]

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Abstract

We study n-ary commutative superalgebras and L_∞-algebras that possess a skew-symmetric invariant form, using the derived bracket formalism. This class of superalgebras includes for instance Lie algebras and their n-ary generalizations, commutative associative and Jordan algebras with an invariant form. We give a classification of anti-commutative m-dimensional $(m-3)$-ary algebras with an invariant form, and a classification of real simple m-dimensional Lie $(m-3)$-algebras with a positive definite invariant form up to isometry. Furthermore, we develop the Hodge Theory for L_∞-algebras with a symmetric invariant form, and we describe quasi-Frobenius structures on skew-symmetric n-ary algebras.

1 Introduction

`Derived bracket formalism.` The derived bracket approach was successfully used in different areas of mathematics: in Poisson geometry, in the theory of Lie algebroids and Courant algebroids, BRST formalism, in the theory of Loday algebras and different types of Drinfeld Doubles. For detailed introduction we recommend a beautiful survey of Y. Kosmann-Schwarzbach [KoSch1].

The idea of the formalism is the following. *One fixes an algebra L, usually a Lie superalgebra, and constructs another multiplication on the same vector space (or some subspace) using derivations of L and the (iterated) multiplication in L. One obtains a class of new algebras, which properties can be studied using original algebra L.* For example, using this formalism we can obtain all Poisson structures on a manifold M from the canonical Poisson algebra on T^*M as was shown by Th. Voronov in [Vor3]. Voronov's idea allows A. Cattaneo and M. Zambon [CZ] to introduce a unified approach to the reduction of Poisson manifolds. Another example was suggested in [Vor1] and [Vor2], where a series of strongly homotopy algebras was obtained from a given Lie superalgebra.

We use this formalism to study n-ary commutative superalgebras with an invariant skew-symmetric form. More precisely, consider a vector superspace V with

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a non-degenerate even skew-symmetric form $(\,,\,)$. There exists a natural Lie superalgebra structure on $S^*(V)$, where $S^*(V)$ is the symmetric power of V. The main observation is that *we get all commutative n-ary and strongly homotopy superalgebras on V with the invariant skew-symmetric form $(\,,\,)$. In other words, the property of these n-ary superalgebras having an invariant skew-symmetric form is encoded by the Lie superalgebra $S^*(V)$.* The observation that using the superalgebra $S^*(V)$ we can obtain all Lie algebras with an invariant symmetric form was made by B. Kostant and S. Sternberg in [KS]. The superalgebra $S^*(V)$ was also used in Poisson Geometry to study for instance Lie bialgebras and Drinfeld Doubles, see [KoSch1, KoSch2], [LR] and others.

`Multiple generalizations of Lie algebras.` Using the derived bracket formalism we can study n-ary commutative superalgebras with a skew-symmetric invariant form. This class of superalgebras includes for instance different n-ary generalizations of a Lie algebra with a symmetric invariant form. First of all let us give a short review of such generalizations.

Multiple generalizations arise usually from different readings of the Jacobi identity. For example, the Jacobi identity for a Lie algebra is equivalent to the statement that all adjoint operators are derivations of this Lie algebra. If we use this point of view for the n-ary case we come to the notion of a *Filippov n-algebra* [Fil]. V.T. Filippov considered alternating n-ary algebras A satisfying the following Jacobi identity:

$$\{a_1, \ldots, a_{n-1}, \{b_1, \ldots, b_n\}\} = \sum \{b_1, \ldots, b_{i-1}\{a_1, \ldots, a_{n-1}, b_i\}, \ldots, b_n\}, \qquad (1)$$

where $a_i,\, b_j \in A$. In other words, the operators $\{a_1, \ldots, a_{n-1}, -\}$ are derivations of the n-ary bracket $\{b_1, \ldots, b_n\}$. Such algebras appear naturally in Nambu mechanics [Nam] in the contecst of Nambu-Poisson manifolds, in supersymmetric gravity theory and in supersymmetric gauge theories, the Bagger-Lambert-Gustavsson Theory, see [AI].

Another natural n-ary generalization of the Jacobi identity has the following form:

$$\sum (-1)^{(I,J)} \{\{a_{i_1}, \ldots, a_{i_n}\}, a_{j_1}, \ldots, a_{j_{n-1}}\} = 0, \qquad (2)$$

where the sum is taken over all ordered unshuffle multi-indexes $I = (i_1, \ldots, i_n)$ and $J = (j_1, \ldots, j_{n-1})$ such that (I, J) is a permutation of $(1, \ldots, 2n-1)$. We will call such algebras *Lie n-algebras*. This type of n-ary algebras was considered for instance by P. Michor and A. Vinogradov in [MV] and by P. Hanlon and M.L. Wachs [HW]. The homotopy case was studied in [SS] in context of the Schlesinger-Stasheff homotopy algebras and L_∞-algebras. Such algebras are related to the Batalin-Fradkin-Vilkovisky theory and to the string field theory, see [LSt]. In [VV1] A.M. Vinogradov and M.M. Vinogradov proposed a three-parameter family

of n-ary algebras such that for some n the above discussed structures appear as particular cases.

The theory of Filippov n-algebras is relatively well-developed. For instance, there is a classification of simple real and complex Filippov n-algebras and an analog of the Levi decomposition [Ling]. W.X. Ling in [Ling] proved that there exists only one simple finite-dimensional Filippov n-algebra over an algebraically closed field of characteristic 0 for any $n > 2$. The simple Filippov n-superalgebras in the finite and infinite dimensional case were studied in [CK]. It was shown there that there are no simple linearly compact Filippov n-superalgebras which are not Filippov n-algebras, if $n > 2$, and a classification of linearly compact Filippov n-algebras was given.

In this paper we give a classification of anti-commutative m-dimensional $(m-3)$-ary algebras with a symmetric invariant form over $\mathbb{R}$ and $\mathbb{C}$ up to isometry in terms of coadjoint orbits of the Lie group $\mathrm{SO}(V)$. In the real positive definite case we give a classification of simple algebras of this type. Our result can be formulated as follows: *almost all real anti-commutative m-dimensional $(m-3)$-ary algebras with a symmetric invariant positive definite form are simple. The exceptional cases are: the trivial $(m-3)$-ary algebra and the $(m-3)$-ary algebras that corresponds to decomposible element.* We also give a classification of real (simple) m-dimensional Lie $(m-3)$-algebras with a symmetric invariant positive definite form.

Hodge decomposition for real strongly homotopy algebras. A definition of a strongly homotopy Lie algebra (or L_∞-algebra or sh-algebra) was given by Lada and Stasheff in [LSt]. For more about strongly homotopy algebras see also [LM], [Vor1], [Vor2]. Another result of our paper is a Hodge Decomposition for real metric pure odd strongly homotopy algebras. An observation here is that we can obtain easily such kind of decomposition using the derived bracket formalism.

We can also use this formalism to define the Hodge operator on a Riemannian compact oriented manifold M. Indeed, in this case there exists the metric on cotangent space T^*M that is induced by Riemannian metric on the tangent space TM. Then we can define a Poisson bracket on $\bigwedge T^*M$, see [Roy], and repeat the construction of the Hodge operator given in the present paper.

Quasi-Frobenius structures. We conclude our paper with a description of quasi-Frobenius structures on anti-commutative n-ary algebras. Our result is as follows. *Assume that n is even. There is a one-to-one correspondence between quasi-Frobenius structures on an anti-commutative n-ary algebra and maximal isotropic subalgebras in T_0^*-extension on this algebra.*

2 Commutative n-ary superalgebras with an invariant skew-symmetric form

2.1 Main definitions

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a finite dimensional $\mathbb{Z}_2$-graded vector space over the field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. If $a \in V$ is a homogeneous element, we denote by $\bar{a} \in \mathbb{Z}_2$ the parity of a. As usual we assume that elements in $\mathbb{K}$ are even. Recall that a bilinear form $(\, ,)$ on V is called *even* (or *odd*) if the corresponding linear map $V \otimes V \to \mathbb{K}$ is even (or odd). A bilinear form is called *skew-symmetric* if $(a, b) = -(-1)^{\bar{a}\bar{b}}(b, a)$ for any homogeneous elements $a, b \in V$.

Definition 1. • An *n-ary superalgebra structure* on V is an n-linear map

$$V \times \cdots \times V \longrightarrow V,$$
$$(a_1, \ldots, a_n) \mapsto \{a_1, \ldots, a_n\}.$$

• An n-ary superalgebra structure is called *commutative* if

$$\{a_1, \ldots, a_i, a_{i+1}, \ldots, a_n\} = (-1)^{\bar{a}_i \bar{a}_{i+1}} \{a_1, \ldots, a_{i+1}, a_i, \ldots, a_n\} \tag{3}$$

for any homogeneous $a_i, a_{i+1} \in V$.

• A commutative n-ary superalgebra structure is called *invariant with respect to the form* $(\, ,)$ if the following holds:

$$(a_0, \{a_1, \ldots, a_n\}) = (-1)^{\bar{a}_0 \bar{a}_1}(a_1, \{a_0, a_2, \ldots, a_n\}) \tag{4}$$

for any homogeneous $a_i \in V$.

We will write *a commutative invariant n-ary superalgebra structure* or *a commutative invariant n-ary superalgebra* as a shorthand for *a commutative n-ary superalgebra structure on V that is invariant with respect to the form* $(\, ,)$.

Example 1. The class of commutative invariant n-ary superalgebras includes for instance the following algebras.

• *Anti-commutative algebras on $V = V_{\bar{1}}$ with an invariant symmetric form.* Indeed, in this case the conditions (3) and (4) are equivalent to the following conditions:

$$\{a, b\} = -\{b, a\}, \quad (\{a, b\}, c) = (a, \{b, c\}). \tag{5}$$

In particular, all Lie algebras with an invariant symmetric form are of this type.

• *Commutative algebras on $V = V_{\bar{0}}$ with an invariant skew-symmetric form.* In this case from (3) and (4) it follows:

$$\{a, b\} = \{b, a\}, \quad (\{a, b\}, c) = -(a, \{b, c\}). \tag{6}$$

In particular, commutative associative and Jordan algebras with an invariant skew-symmetric form are of this type.

- *Anti-commutative n-ary algebras on $V = V_{\bar{1}}$ with an invariant symmetric form.* In this case the condition (4) is equivalent to the following condition:

$$(y, \{x_1, \ldots, x_{n-1}, z\}) = (-1)^n(\{y, x_1, \ldots, x_{n-1}\}, z)$$

that is more familiar for physicists. In particular, anti-commutative n-ary algebras satisfying (1) with an invariant symmetric form are of this type. Such algebras are used in the Bagger-Lambert-Gustavsson model (BLG-model), see [AI] for details.

Remark. For a commutative algebra usually one considers the following invariance condition: $(\{a, b\}, c) = (a, \{b, c\})$. If in addition we assume that the form $(\,,)$ is skew-symmetric and non-degenerate, we obtain $2(ab, c) = 0$ for all $a, b, c \in V$, therefore $ab = 0$. In our case we do not have such additional restrictive relations.

2.2 Derived bracket and commutative invariant n-ary superalgebras

Let V be as above. We denote by $S^n V$ the n-th symmetric power of V and we put $S^* V = \bigoplus_n S^n V$. The superspace $S^* V$ possesses a natural structure $[\,,]$ of a Poisson superalgebra. It is defined by the following formulas:

$$[x, y] := (x, y), \quad x, y \in V;$$

$$[v, w_1 \cdot w_2] := [v, w_1] \cdot w_2 + (-1)^{vw_1} w_1 \cdot [v, w_2],$$

$$[v, w] = -(-1)^{vw}[w, v],$$

where v, w, w_i are homogeneous elements in $S^* V$. One can show that the multiplication $[\,,]$ satisfies the graded Jacobi identity:

$$[v, [w_1, w_2]] = [[v, w_1], w_2] + (-1)^{\bar{v}\bar{w}_1}[w_1, [v, w_2]].$$

This Poisson superalgebra is well-defined. Indeed, we can repeat the argument from [KS, Page 65] for vector superspaces. The idea is to show that this superalgebra is induced by the Clifford superalgebra corresponding to V and $(\,,)$.

Let us take any element $\mu \in S^{n+1} V$. Then we can define an n-ary superalgebra structure on V in the following way:

$$\{a_1, \ldots, a_n\} := [a_1, [\ldots, [a_n, \mu] \ldots]], \quad a_i \in V. \tag{7}$$

We will denote the corresponding superalgebra by (V, μ) and we will call the element μ the *derived potential* of (V, μ). The n-ary superalgebras of type (V, μ) have the following two properties:

- The multiplication (7) is *commutative*. (This was noticed in [Vor1].) Indeed, using Jacobi identity for S^*V we have:

$$[a_1, [a_2, \ldots, [a_n, \mu] \ldots]] = [[a_1, a_2], [\ldots, [a_n, \mu] \ldots]] +$$
$$(-1)^{\bar{a}_1 \bar{a}_2} [a_2, [a_1, \ldots, [a_n, \mu] \ldots]] = (-1)^{\bar{a}_1 \bar{a}_2} [a_2, [a_1, \ldots, [a_n, \mu] \ldots]].$$

We used the fact that $[[a_1, a_2], [\ldots, [a_n, \mu] \ldots]] = 0$, because $[a_1, a_2] \in \mathbb{K}$. Similarly we can prove the commutativity relation for other a_i.

- The n-ary superalgebra structure (7) is invariant. Indeed,

$$(a_0, \{a_1, \ldots, a_n\}) = [a_0, [a_1, [a_2, \ldots, [a_n, \mu] \ldots]]] =$$
$$(-1)^{\bar{a}_0 \bar{a}_1} [a_1, [a_0, [a_2, \ldots, [a_n, \mu] \ldots]]] = (-1)^{\bar{a}_0 \bar{a}_1} (a_1, \{a_0, a_2, \ldots, a_n\}).$$

We conclude this section with the following observation.

Proposition 1. *Assume that V is finite dimensional and $(\, ,)$ is non-degenerate. Any commutative invariant n-ary superalgebra structures can be obtained by construction (7).*

Proof. Denote by $\mathcal{A}_n$ the vector space of commutative invariant n-ary superalgebra structures on V and by $\mathcal{L}_{n+1}$ the vector space of symmetric $(n+1)$-linear maps from V to $\mathbb{K}$. Clearly, $\dim \mathcal{L}_{n+1} = \dim S^{n+1}V$. Since $(\, ,)$ is non-degenerate, Formula (7) defines an injective linear map $S^{n+1}V \to \mathcal{A}_n$. We can also define an injective linear map $\mathcal{A}_n \to \mathcal{L}_{n+1}$ in the following way:

$$\mathcal{A}_n \ni \mu \longmapsto L_\mu \in \mathcal{L}_{n+1}, \quad L_\mu(a_1, \ldots, a_{n+1}) = (a_1, \mu(a_2, \ldots, a_{n+1})).$$

Note that L_μ is symmetric since μ defines an invariant superalgebra structure. Summing up, we have the following sequence of injective maps or isomorphisms:

$$S^{n+1}V \hookrightarrow \mathcal{A}_n \hookrightarrow \mathcal{L}_{n+1} \simeq S^{n+1}V.$$

Since V is finite dimensional, we get $S^{n+1}V \simeq \mathcal{A}_n.\square$

3 Examples of commutative invariant n-ary superalgebras

Usually one studies superalgebras with an invariant form in the following way. One considers for example a Lie algebra or a Jordan algebra and assumes that the multiplication in the algebra satisfies the following additional condition: it is invariant with respect to a non-degenerate (skew)-symmetric form. The derived bracket formalism permits to express for instance Jacobi, Filippov and Jordan identities in terms of derived potentials and the Poisson bracket on S^*V. In this case the additional invariance condition is fulfilled automatically.

3.1 Strongly homotopy Lie algebras with an invariant skew-symmetric form

We follow Th. Voronov [Vor1] in conventions concerning L_∞-algebras. We set $I^k := (i_1, \ldots, i_k)$ and $J^l := (j_1, \ldots, j_l)$, where $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_l$. We denote

$$a_{I^k} := (a_{i_1}, \ldots, a_{i_k}), \quad a_{J^l} := (a_{j_1}, \ldots, a_{j_l}) \quad \text{and} \quad a^s := (a_1, \ldots, a_s),$$

where $a_s \in V$. We put $[a_{I^k}, \mu] := [a_{i_1}, \ldots [a_{i_k}, \mu]]$ and $[a^s, \mu] := [a_1, \ldots [a_s, \mu]]$, where $\mu \in S^*V$.

Definition 2. A vector superspace V with a sequence of odd commutative n-linear maps μ_n, where $n \geq 0$, is called an L_∞-*algebra* if the following generalized Jacobi identities hold:

$$\sum_{k+l=n} \sum_{(I^k, J^l)} (-1)^{(I^k, J^l)} \mu_{l+1}(a_{I^l}, \mu_k(a_{J^k})) = 0, \quad n \geq 0. \tag{8}$$

Here (I^k, J^l) is a unshuffle permutation of $(1, \ldots, n)$ and $(-1)^{(I^k, J^l)}$ is the sign obtained using the sign rule for the permutation (I^k, J^l) of homogeneous elements $a_1, \ldots, a_n \in V$.

Definition 3. An L_∞-algebra structure $(\mu_n)_{n \geq 0}$ on V is called *invariant* if all μ_n are invariant in the sense of Definition 1.

The following statement can be deduced from [Vor1, Theorem 1] and Proposition 1. For completeness we give here a proof in our notations and agreements.

Proposition 2. *Invariant L_∞-algebra structures on V are in one-to-one correspondence with odd elements $\mu \in S^*(V)$ such that $[\mu, \mu] \in S^0V = \mathbb{K}$.*

Proof. Our objective is to show that $[\mu, \mu] \in \mathbb{K}$ is equivalent to (8) together with the invariance condition. Let us take any odd element $\mu = \sum_k \mu_k \in S^*V$, where $\mu_k \in S^{k+1}V$. The condition $[\mu, \mu] \in \mathbb{K}$ is equivalent to the following conditions

$$\sum_{k+l=n} [\mu_l, \mu_k] = 0, \quad n \geq 1.$$

Note that $[\mu_0, \mu_0]$ is always an element is $\mathbb{K}$. This is equivalent to:

$$\sum_{k+l=n} [a^{n-1}, [\mu_l, \mu_k]] = 0, \quad n \geq 1.$$

Furthermore, we have:

$$[a^{n-1}, [\mu_l, \mu_k]] = \sum_{(I^l, J^{k-1})} (-1)^{(I^l, J^{k-1}) + \bar{a}_{J^{k-1}}} [[a_{I^l}, \mu_l], [a_{J^{k-1}}, \mu_k]] +$$

$$\sum_{(I^{l-1}, J^k)} (-1)^{(I^{l-1}, J^k) + \bar{a}_{J^k}} [[a_{I^{l-1}}, \mu_l], [a_{J^k}, \mu_k]],$$

where $\bar{a}_{J^{k-1}}$ and $\bar{a}_{J^k}$ are the parities of $a_{J^{k-1}}$ and a_{J^k}, respectively. Denote by $\tilde{\mu}_s$ the s-linear map defined by $\tilde{\mu}_s(a_1, \ldots, a_s) := [a^s, \mu_s]$. We get:

$$[[a_{I^l}, \mu_l], [a_{J^{k-1}}, \mu_k]] = \tilde{\mu}_k(\tilde{\mu}_l(a_{I^l}), a_{J^{k-1}});$$

$$[[a_{I^{l-1}}, \mu_l], [a_{J^k}, \mu_k]] = (-1)^{(\bar{a}_{J^{l-1}} - 1)(\bar{a}_{J^k} - 1) + 1} \tilde{\mu}_l(\tilde{\mu}_k(a_{I^k}), a_{J^{l-1}}).$$

Further,

$$[a^{n-1}, [\mu_l, \mu_k]] = \sum_{(I^l, J^{k-1})} (-1)^{(I^l, J^{k-1}) + \bar{a}_{J^{k-1}}} \tilde{\mu}_k(\tilde{\mu}_l(a_{I^l}), a_{J^{k-1}}) +$$

$$\sum_{(I^k, J^{l-1})} (-1)^{(J^k, I^{l-1}) + \bar{a}_{J^{l-1}}} \tilde{\mu}_l(\tilde{\mu}_k(a_{I^k}), a_{J^{l-1}}) =$$

$$\sum_{(J^{k-1}, I^l)} (-1)^{(J^{k-1}, I^l)} \tilde{\mu}_k(a_{J^{k-1}}, \tilde{\mu}_l(a_{I^l})) +$$

$$\sum_{(I^{l-1}, J^k)} (-1)^{(I^{l-1}, J^k)} \tilde{\mu}_l(a_{I^{l-1}}, \tilde{\mu}_k(a_{J^k})).$$

Using the equolities:

$$\sum_{k+l=n} \sum_{(J^{k-1}, I^l)} (-1)^{(J^{k-1}, I^l)} \tilde{\mu}_k(a_{J^{k-1}}, \tilde{\mu}_l(a_{I^l})) =$$

$$\sum_{k'+l=n-1} \sum_{(J^{k'}, I^l)} (-1)^{(J^{k'}, I^l)} \tilde{\mu}_{k'+1}(a_{J^{k'}}, \tilde{\mu}_l(a_{I^l}));$$

$$\sum_{k+l=n} \sum_{(I^{l-1}, J^k)} (-1)^{(I^{l-1}, J^k)} \tilde{\mu}_l(a_{I^{l-1}}, \tilde{\mu}_k(a_{J^k})) =$$

$$\sum_{k+l'=n-1} \sum_{(I^{l'}, J^k)} (-1)^{(I^{l'}, J^k)} \tilde{\mu}_{l'+1}(a_{I^{l'}}, \tilde{\mu}_k(a_{J^k}))$$

we see that

$$[a^{n-1}, \sum_{k+l=n} [\mu_l, \mu_k]] = 2 \sum_{k'+l=n-1} \sum_{(J^{k'}, I^l)} (-1)^{(J^{k'}, I^l)} \tilde{\mu}_{k'+1}(a_{J^{k'}}, \tilde{\mu}_l(a_{I^l})). \qquad (9)$$

Therefore, $[a^{n-1}, \sum_{k+l=n} [\mu_l, \mu_k]] = 0$ is equivalent to the generalized $(n-1)$-Jacobi identity for the invariant L_∞-algebra $\{\tilde{\mu}_s\}$. Conversely, if an invariant L_∞-algebra $\{\tilde{\mu}_s\}$ is given, its derived potential $\mu = \sum_s \mu_s$, where $\mu_s \in S^{s+1}V$ corresponds to $\tilde{\mu}_s$ by Proposition 1, must satisfy the condition $[\mu, \mu] \in \mathbb{K}$. $\square$

Corollary. *Assume that $V = V_{\bar{1}}$ and n is even. Anti-commutative invariant n-ary algebra structures, where $n > 0$, on V satisfying the Jacobi identity (2) are in one-to-one correspondence with elements $\mu \in S^{n+1}(V)$ such that $[\mu, \mu] = 0$.*

Proof. In this case the equation 9 has the form:

$$[a^{2n-1}, [\mu, \mu]] = 2 \sum_{(I,J)} (-1)^{(I,J)} \tilde{\mu}(a^I, \tilde{\mu}(a^J)).$$

Here $I = (i_1, \ldots, i_{n-1})$ and $J = (j_1, \ldots, j_n)$ such that $i_1 < \cdots < i_{n-1}, j_1 < \cdots < j_n$ and $I \cup J = \{1, \ldots, 2n-1\}$. Since n is even we have:

$$\sum_{(I,J)} (-1)^{(I,J)} \tilde{\mu}(a^I, \tilde{\mu}(a^J)) = -\sum_{(J,I)} (-1)^{(J,I)} \tilde{\mu}(\tilde{\mu}(a^J), a^I),$$

and the equality

$$\sum_{(J,I)} (-1)^{(J,I)} \tilde{\mu}(\tilde{\mu}(a^J), a^I) = 0$$

is equivalent to the Jacobi identity (2) for n-ary algebra structure $\tilde{\mu}$.$\square$

3.2 Filippov algebras with an invariant symmetric form

The class of commutative symmetric superalgebras includes n-ary algebras introduced by Filippov [Fil].

Definition 4. • An anti-commutative n-ary algebra is called a *Filippov n-algebra* if it satisfies the Jacobi identity (1).

   • We say that a Filippov n-algebra V has an *invariant symmetric form* $(\,,)$ if the following holds

$$(y, \{x_1, \ldots, x_{n-1}, z\}) = (-1)^n (\{y, x_1, \ldots, x_{n-1}\}, z)$$

for any $x_i, y, z \in V$.

   Filippov n-algebras with an invariant form are described in the following proposition. The idea of the proof we borrow from [VV1].

Proposition 3. *Assume that $V = V_{\bar{1}}$. Invariant Filippov n-algebra structures on V are in one-to-one correspondence with elements $\mu \in S^{n+1}V$ such that the following equation holds:*

$$[\mu_{a^{n-1}}, \mu] = 0$$

for all $a^{n-1} = (a_1, \ldots, a_{n-1})$ and $\mu_{a^{n-1}} := [a_1, \ldots, [a_{n-1}, \mu]]$.

Proof. We need to show that $[\mu_{a^{n-1}}, \mu] = 0$ is equivalent to (1) together with the invariance condition. Let us take $b_1 \ldots, b_n \in V$. We have:

$$[\mu_{a^{n-1}}, [b_1, \ldots [b_n, \mu]]] = \sum_{i=1}^{n} [b_1, \ldots [[\mu_{a^{n-1}}, b_i], \ldots [b_n, \mu]]] + [b_1, \ldots [b_n, [\mu_{a^{n-1}}, \mu]]]$$

Further, using (7), we get:

$$[\mu_{a^{n-1}}, [b_1, \ldots [b_n, \mu]]] = -\{\{b_1, \ldots, b_n\}, a_1 \ldots, a_{n-1}\} =$$
$$(-1)^n \{a_1, \ldots, a_{n-1}, \{b_1, \ldots, b_n\}\};$$

and

$$[b_1, \ldots [[\mu_{a^{n-1}}, b_i], \ldots [b_n, \mu]]] = -\{b_1 \ldots, b_{i-1}, \{b_i, a_1 \ldots, a_{n-1}\}, b_{i+1}, \ldots, b_n\} =$$
$$(-1)^n \{b_1 \ldots, b_{i-1}, \{a_1 \ldots, a_{n-1}, b_i\}, b_{i+1}, \ldots, b_n\};$$

Hence, we have:

$$\{a_1, \ldots, a_{n-1}, \{b_1, \ldots, b_n\}\} = \sum_{i=1}^{n} \{b_1 \ldots, b_{i-1}, \{a_1 \ldots, a_{n-1}, b_i\}, b_{i+1}, \ldots, b_n\} +$$
$$(-1)^n [b_1, \ldots [b_n, [\mu_{a^{n-1}}, \mu]]].$$

Therefore, the condition $[b_1, \ldots [b_n, [\mu_{a^{n-1}}, \mu]]] = 0$ is equivalent to (1) together with the invariance condition. The proof is complete. $\square$

3.3 Jordan algebras with a skew-symmetric invariant form

First of all let us recall the definition of a Jordan algebra.

Definition 5. A *Jordan algebra* is a commutative algebra such that the multiplication satisfies the following condition:

$$(xy)(xx) = x(y(xx)).$$

Definition 6. We say that a Jordan algebra V has an *invariant skew-symmetric form* $(\,,\,)$ if the following holds:

$$(ab, c) = -(a, bc)$$

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for any $a, b, c \in V$.

A description of invariant Jordan algebrs structures on V is given in the following proposition.

Proposition 4. *Let V be a pure even vector space with a non-degenerate skew-symmetric form $(\,,)$. Invariant Jordan algebra structures on V are in one-to-one correspondence with elements $A \in S^3 V$ such that the following identity holds:*

$$[A_x, A_{[A_x, x]}] = 0,$$

where $A_x = [x, A]$.

Proof. By Proposition 1 any invariant commutative algebra structure $(x, y) \mapsto xy$ on V can be obtained by the derived bracket construction. Denote by A its derived potential. In other words, we have:

$$xy = [x, [y, A]].$$

Further,

$$(xy)(xx) = [[y, A_x], [[x, A_x], A]]; \quad x(y(xx)) = -[A_x, [y, [[x, A_x], A]]].$$

Using the Jacobi identity for the Poisson algebra $S^* V$, we get:

$$[A_x, [y, [[x, A_x], A]]] = [[A_x, y], [[x, A_x], A]] + [y, [A_x, [[x, A_x], A]]].$$

We see that this equation is equivalent to

$$-x(y(xx)) = -(xy)(xx) + [y, [A_x, [[x, A_x], A]]].$$

Hence, the algebra (V, A) is Jordan if and only if

$$[y, [A_x, [[x, A_x], A]]] = 0$$

for all $x, y \in V$. The last condition is equivalent to

$$[A_x, [[x, A_x], A]] = 0$$

for all $x \in V$. $\square$

3.4 Associative algebras with a skew-symmetric invariant form

Proposition 5. *Let V be a pure even vector space with a non-degenerate skew-symmetric form $(\,,\,)$. Invariant associative algebra structures on V are in one-to-one correspondence with elements $\mu \in S^3 V$ such that the following identity holds:*

$$[\mu_a, \mu_b] = 0$$

for all $a, b \in V$. Here $\mu_x = [x, \mu]$ for $x \in V$.

Proof. Let us use the notation: $a \circ b := [a, [b, \mu]]$. We have to show that the associativity relation for $\circ$ together with the invariance condition is equivalent to $[\mu_a, \mu_b] = 0$ for all $a, b \in V$. Indeed,

$$a \circ (b \circ c) = [a, [[b, [c, \mu]]\mu]] = -[a, [\mu, [b, [c, \mu]]]] = -[\mu_a, [b, [c, \mu]]] =$$
$$-[[\mu_a, b], [c, \mu]] - [b, [\mu_a, \mu_c]] = [[b, \mu_a], [c, \mu]] - [b, [\mu_a, \mu_c]] =$$
$$(b \circ a) \circ c - [b, [\mu_a, \mu_c]].$$

Therefore, the associativity relation for $\circ$ together with the invariance condition and the equality $[\mu_a, \mu_c] = 0$ for all $a, c \in V$ are equivalent. $\square$

4 Hodge operator and its applications

4.1 $*$-operator and n-ary algebras

Let V be a pure odd vector space of dimension m with a non-degenerate skew-symmetric even bilinear form $(\,,\,)$. Recall that means that $(a, b) = (b, a)$ for all $a, b \in V$. Let us choose a normalized orthogonal basis (e_i) of V. Denote by $L := e_1 \ldots e_m$ the top form corresponding to the chosen basis. We define the operator $* : S^p V \to S^{m-p} V$ by the following formula:

$$*(x_1 \ldots x_p) = [x_1, [\ldots [x_p, L]]]. \tag{10}$$

In particular, we have:

$$*(e_{i_1} \ldots e_{i_p}) = [e_{i_1}, [\ldots [e_{i_p}, L]]] = (-1)^\sigma e_{j_1} \ldots e_{j_{m-p}},$$

where $\sigma(1, \ldots, m) = (i_p, \ldots, i_1, j_1, \ldots, j_{m-p})$. Clearly, this definition depends only on orientation of V and on the bilinear form $(\,,\,)$. Note that $* : S^p V \to S^{m-p} V$ is an isomorphism for all p. This follows for example from the following formula:

$$* * (e_{i_1} \ldots e_{i_p}) = (-1)^{\frac{m(m-1)}{2}} e_{i_1} \ldots e_{i_p}.$$

The following well-known result we can easily prove using derived bracket formalism:

Proposition 6. *The vector space $\mathfrak{so}(V)$ of linear operators preserving the form $(\,,)$ is isomorphic to $S^2(V)$.*

Proof. The isomorphism is given by the formula $w \longmapsto \operatorname{ad} w$, where $w \in S^2(V)$ and $\operatorname{ad} w(v) := [w, v]$ for $v \in V$. Indeed, for all $v_1, v_2 \in V$ we have:

$$0 = \operatorname{ad} w([v_1, v_2]) = [w, [v_1, v_2]] = [[w, v_1], v_2] + [v_1, [w, v_2]] =$$
$$([w, v_1], v_2) + (v_1, [w, v_2]).$$

Clearly, this map is injective. We complete the proof observing that the dimensions of $\mathfrak{so}(V)$ and $S^2(V)$ are equal.$\square$

We have seen in previous sections that elements from $S^{n+1}V$ corresponds to invariant n-ary algebra structures on V. The existence of the $*$-operator for $V = V_{\bar{1}}$ leads to the idea that n-ary and $(m-n)$-ary algebras can have some common properties. In particular such algebras have the same algebra of orthogonal derivations.

Definition 7. A *derivation* of an n-ary algebra (V, μ) is a linear map $D : V \to V$ such that
$$D(\{v_1, \ldots, v_n\}) = \sum_j \{v_1, \ldots, D(v_j), \ldots, v_n\}.$$

We denote by $\operatorname{IDer}(\mu)$ the vector space of all derivations of the algebra (V, μ) preserving the form $(\,,)$.

Proposition 7. *Let us take any $w \in S^2(V)$ and $\mu \in S^{n+1}(V)$.*
a. *We have:*
$$\operatorname{IDer}(\mu) = \{w \in S^2(V) \mid \operatorname{ad} w(\mu) = 0\}.$$

b. *The isomorphism $* : S^p(V) \to S^{m-p}(V)$ is equivariant with respect to the natural action of $\mathfrak{so}(V)$ on $S^*(V)$. In particular,*

$$\operatorname{IDer}(\mu) = \operatorname{IDer}(*\mu).$$

Proof. **a.** First of all using the Jacobi identity for S^*V we obtain:

$$\operatorname{ad} w(\{v_1, \ldots, v_p\}) = [w, [v_1, \ldots, [v_n, \mu] \ldots]] = \sum_i^n [v_1, \ldots, [[w, v_i] \ldots, [v_n, \mu]] \ldots] +$$

$$[v_1, \ldots, [v_n, [w, \mu]] \ldots] = \sum_j \{v_1, \ldots, [w, v_j], \ldots, v_n\} + [v_1, \ldots, [v_n, [w, \mu]] \ldots].$$

We see that $\operatorname{ad} w$ is a derivation of (V, μ) if and only if $[w, \mu] = 0$.

b. Let $L = e_1 \ldots e_m$ be as above and $w \in S^2(V)$. We have,

$$*([w, e_{i_1} \ldots e_{i_p}]) = *\left(\sum_{j=1}^{p} e_{i_1} \ldots [w, e_{i_j}] \ldots e_{i_p}\right) =$$

$$\sum_{j=1}^{p} [e_{i_1}, \ldots, [[w, e_{i_j}] \ldots, [e_{i_p}, L]] \ldots].$$

On the other side,

$$[w, *(e_{i_1} \ldots e_{i_p})] = [w, [e_{i_1}, \ldots [e_{i_p}, L]]] = \sum_{j=1}^{p} [e_{i_1}, \ldots, [[w, e_{i_j}] \ldots, [e_{i_p}, L]] \ldots].$$

We use here the fact that $[w, L] = 0$. Therefore, the $*$-operator is $\mathfrak{so}(V)$-equivariant.

Furthermore, assume that $w \in \mathrm{IDer}(\mu)$ or equivalently that $[w, \mu] = 0$. Therefore,

$$[w, *\mu] = *([w, \mu]) = *(0) = 0.$$

Hence, $w \in \mathrm{IDer}(*\mu)$. Conversely, if $w \in \mathrm{IDer}(*\mu)$ then

$$*([w, \mu]) = [w, *\mu] = 0.$$

This finishes the proof. $\square$

4.2 Hodge decomposition for real metric strongly homotopy algebras

4.2.1 Hodge decomposition for a vector space.

In this Subsection we follow Kostant's approach [Kost, Page 332 - 333]. Let W be a finite dimensional vector space with two linear operators d and δ.

Definition 8. [Kostant] Linear maps d and δ are called *disjoint* if the following holds:

1. $\mathrm{d} \circ \delta(x) = 0$ implies $\delta(x) = 0$;

2. $\delta \circ \mathrm{d}(x) = 0$ implies $\mathrm{d}(x) = 0$.

Denote $\mathcal{L} = \delta \circ \mathrm{d} + \mathrm{d} \circ \delta$.

Proposition 8. [Kostant] *Assume that* d *and* δ *are disjoint and* $\mathrm{d}^2 = \delta^2 = 0$. *Then we have* $\mathrm{Ker}(\mathcal{L}) = \mathrm{Ker}(\mathrm{d}) \cap \mathrm{Ker}(\delta)$ *and the direct sum (an analog of a Hodge Decomposition):*

$$W = \mathrm{Im}(\mathrm{d}) \oplus \mathrm{Im}(\delta) \oplus \mathrm{Ker}(\mathcal{L}).$$

In this case the restriction $\pi|_{\mathrm{Ker}(\mathcal{L})}$ of the canonical mapping

$$\pi : \mathrm{Ker}(\mathrm{d}) \to \mathrm{Ker}(\mathrm{d})/\mathrm{Im}(\mathrm{d}) =: H(W, \mathrm{d})$$

is a bijection. In other words $\mathrm{Ker}(\mathcal{L}) \simeq H(W, \mathrm{d})$.$\square$

We will use this Proposition to obtain a Hodge decomposition for metric L_∞-algebras.

4.2.2   Hodge decomposition for real metric L_∞-algebras.

Let V be a pure odd real m-dimensional vector space with a non-degenerate skew-symmetric positive defined form $(\, ,)$. We can define a bilinear product $\langle \, , \rangle$ in S^*V by the following formula:

$$\langle v_1, v_2 \rangle L = \begin{cases} (-1)^{\frac{p(p-1)}{2}} v_1 \cdot (*v_2), & \text{if } v_1, v_2 \in S^p V; \\ 0, & \text{if } v_1 \in S^p V, \ v_2 \in S^q V \text{ and } p \neq q. \end{cases}$$

This bilinear product has the following properties:

Proposition 9. *Let us take $I = (i_1, \ldots, i_p)$ and $J = (j_1, \ldots, j_p)$ such that $i_1 < \cdots < i_p$ and $j_1 < \cdots < j_p$. Denote $e_I := e_{i_1} \cdots e_{i_p}$ and $e_j := e_{j_1} \cdots e_{j_p}$ We have*

$$\langle e_I, e_J \rangle = \begin{cases} 0, & \text{if } I \neq J, \\ 1, & \text{if } I = J. \end{cases}$$

In particular, the pairing $\langle \, , \rangle$ is symmetric and positive definite.

Proof. A straightforward computation.$\square$

Let $\mu \in S^*V$ be any element. Denote by $\mathrm{d} : S^*V \to S^*V$ the linear operator $v \mapsto [\mu, v]$. Let $\mu = \sum_k \mu_k$, where $\mu_k \in S^{k+2}V$, and we put $\mathrm{d}_k := [\mu_k, -]$. Using Hodge $*$-operator we can define the following operator

$$\delta = \sum_k (-1)^{\frac{k(1-k)}{2}} \delta_k,$$

where $\delta_k := * \, \mathrm{d}_k \, *$.

Proposition 10. *Assume that $\mu \in S^*(V)$, d and δ are as above. Then we have*

$$\langle \mathrm{d}(v), w \rangle = -(-1)^{\frac{m(m-1)}{2}} \langle v, \delta(w) \rangle$$

for $v, w \in S^*V$. The operators d and δ are disjoint.

Proof. Let us take $\mu_k \in S^{k+2}V$, $v \in S^{p-k}V$ and $w \in S^pV$. (We assume that $S^rV = \{0\}$ for $r < 0$ and $r > m$, where $m = \dim V$.) Then, $v \cdot *w \in S^{m-k}V$ and we have:

$$[\mu_k, v \cdot *w] \subset [S^{k+2}V, S^{m-k}(V)] = 0.$$

Furthermore,

$$0 = [\mu_k, v \cdot *w] = [\mu_k, v] \cdot *w + (-1)^{\bar{\mu}_k \bar{v}} v \cdot [\mu_k, *w] = [\mu_k, v] \cdot *w +$$
$$(-1)^{\bar{\mu}_k \bar{v} + \frac{m(m-1)}{2}} v \cdot * * [\mu_k, *w] = d_k(v) \cdot *w + (-1)^{\bar{\mu}_k \bar{v} + \frac{m(m-1)}{2}} v \cdot *\delta_k(w),$$

where $d_k(v) = [\mu_k, v]$ and $\delta_k(w) = *[\mu_k, *w]$. Further,

$$d_k(v) \cdot *w = (-1)^{\frac{p(p-1)}{2}} \langle d_k(v), w \rangle L; \quad v \cdot *\delta_k(w) = (-1)^{\frac{(p-k)(p-k-1)}{2}} \langle v, \delta_k(w) \rangle L.$$

Therefore,

$$(-1)^{\frac{p(p-1)}{2}} \langle d_k(v), w \rangle = -(-1)^{\frac{(p-k)(p-k-1)}{2}} (-1)^{\bar{\mu}_k \bar{v} + \frac{m(m-1)}{2}} \langle v, \delta_k(w) \rangle$$

or

$$\langle d_k(v), w \rangle = -(-1)^{\frac{k(1-k)}{2}} (-1)^{\frac{m(m-1)}{2}} \langle v, \delta_k(w) \rangle$$

for all $v \in S^{p-k}V$ and $w \in S^pV$. Note that this equation holds trivially for $v \in S^sV$ and $w \in S^qV$, where $q - s \neq k$. Therefore, we have

$$\langle d_k(v), w \rangle = -(-1)^{\frac{k(1-k)}{2}} (-1)^{\frac{m(m-1)}{2}} \langle v, \delta_k(w) \rangle \tag{11}$$

for all $v, w \in S^*V$ and $\mu_k \in S^{k+2}V$.

Let us take any $\mu \in S^*(V)$. Then $\mu = \sum_k \mu_k$, where $\mu_k \in S^k(V)$. Therefore, d and δ also possess corresponding decomposition: $d = \sum_k d_k$ and $\delta = \sum_k (-1)^{\frac{k(1-k)}{2}} \delta_k$, where $d_k = [\mu_k, -]$ and $\delta_k = *d_k*$. Using (11), we get for any $v, w \in S^*V$:

$$\langle d(v), w \rangle = \sum_k \langle d_k(v), w \rangle = -\sum_k (-1)^{\frac{k(1-k)}{2}} (-1)^{\frac{m(m-1)}{2}} \langle v, \delta_k(w) \rangle =$$
$$-(-1)^{\frac{m(m-1)}{2}} \sum_k (-1)^{\frac{k(1-k)}{2}} \langle v, \delta_k(w) \rangle = -(-1)^{\frac{m(m-1)}{2}} \langle v, \delta(w) \rangle.$$

The first statement is proven.

Let us show that $d \circ \delta(v) = 0$ implies $\delta(v) = 0$, i.e. the operators d and δ are disjoint. (This argument we borrow from [Kost].) Indeed,

$$0 = \langle d \circ \delta(v), v \rangle = -(-1)^{\frac{m(m-1)}{2}} \langle \delta(v), \delta(v) \rangle.$$

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The pairing $\langle\,,\rangle$ is positive definite, hence $\delta(v) = 0$. Analogously we can show that $\delta \circ \mathrm{d}(v) = 0$ implies $\mathrm{d}(v) = 0$.$\square$

Assume that (V, μ) is an L_∞-algebra. By Proposition 2 this means that μ is an odd element and $[\mu, \mu] \in \mathbb{K}$. Denote by $H(V, \mu)$ the cohomology group of (V, μ). By definition $H(V, \mu) := \mathrm{Ker}(\mathrm{d})/\mathrm{Im}(\mathrm{d})$, where $\mathrm{d} = [\mu, -] : S^*V \to S^*V$. (Clearly, $\mathrm{d}^2 = \delta^2 = 0$.) The main result of this section is the following theorem.

Theorem 1. [Hodge decomposition for real metric L_∞-algebras] *Let $\mu \in S^*(V)$ be a real metric L_∞-algebra structure on V and d and δ be as above. Then we have a direct sum decomposition:*

$$V = \mathrm{Im}(\mathrm{d}) \oplus \mathrm{Im}(\delta) \oplus \mathrm{Ker}(\mathcal{L}),$$

where $\mathcal{L} = \delta \circ \mathrm{d} + \mathrm{d} \circ \delta$, and $\mathrm{Ker}(\mathcal{L}) \simeq H(V, \mu)$.

Proof. The statement follows from Propositions 8 and 10 $\square$.

5 Filippov and Lie m-dimensional invariant $(m-3)$-algebras

5.1 Anti-commutative m-dimensional invariant $(m-3)$-ary algebras and coadjoint orbits

In this section we will classify all $(m-3)$-ary anti-commutative algebra structures on V, where $\dim V = m$, up to orthogonal isomorphism in terms of coadjoint orbits. Let again V be a pure odd vector space with an even non-degenerate skew-symmetric form $(\,,\,)$. That is $V = V_{\bar{1}}$ and $(a, b) = (b, a)$ for all $a, b \in V$. As usual we denote by $\mathrm{O}(V)$ the Lie group of all invertible linear operators on V that preserve the form $(\,,\,)$ and by $\mathrm{SO}(V)$ the subgroup of $\mathrm{O}(V)$ that contains all operators with the determinant $+1$. We have $\mathfrak{so}(V) = \mathrm{Lie}\,\mathrm{O}(V) = \mathrm{Lie}\,\mathrm{SO}(V)$.

Definition 9. Two n-ary algebra structures $\mu, \mu' \in S^*V$ on V are called *isomorphic* if there exists $\varphi \in \mathrm{SO}(V)$ such that

$$\varphi(\{v_1, \ldots, v_n\}_\mu) = \{\varphi(v_1), \ldots, \varphi(v_n)\}_{\mu'}$$

for all $v_i \in V$. Here we denote by $\{\ldots\}_\nu$ the multiplication on V corresponding to the algebra structure ν.

Sometimes we will consider isomorphism of n-ary algebra structures up to $\varphi \in \mathrm{O}(V)$. We need the following two lemmas:

Lemma 1. *Let us take $\varphi \in \mathrm{O}(V)$ and $w, v \in S^*V$. Then, $\varphi([w, v]) = [\varphi(w), \varphi(v)]$.*

Proof. It follows from the following two facts:

- $(\varphi(w), \varphi(v)) = (w, v)$, if $w, v \in V$;

- $\varphi(w \cdot v) = \varphi(w) \cdot \varphi(v)$ for all $w, v \in S^*V$. $\square$

Lemma 2. *Two n-ary algebra structures μ and μ', where $\mu, \mu' \in S^{n+1}V$, are isomorphic if and only if there exists $\varphi \in \mathrm{SO}(V)$ such that $\varphi(\mu) = \mu'$. In other words, two n-ary algebra structures are isomorphic if and only if they are in the same orbit of the action $\mathrm{SO}(V)$ on $S^{n+1}V$.*

Proof. From Lemma 1 it follows that

$$\varphi(\{v_1, \ldots, v_n\}_\mu) = \{\varphi(v_1), \ldots, \varphi(v_n)\}_{\varphi(\mu)}.$$

Therefore, n-ary algebra structures μ and $\varphi(\mu)$ are isomorphic.

Conversely, if μ and μ' are isomorphic and $\varphi \in \mathrm{SO}(V)$ is an isomorphism then from the definition it follows that:

$$\{\varphi(v_1), \ldots, \varphi(v_n)\}_{\varphi(\mu)} = \{\varphi(v_1), \ldots, \varphi(v_n)\}_{\mu'}$$

for all $v_i \in V$. Therefore, $\varphi(\mu) = \mu'$. $\square$

Theorem 2. *Assume that $\dim V = m$. Classes of isomorphic real or complex invariant $(m-3)$-ary algebra structures on V are in one-to-one correspondence with coadjoint orbits of the Lie group $\mathrm{SO}(V)$.*

Proof. Note that in the case of the Lie group $\mathrm{SO}(V)$ the adjoint and coadjoint actions are equivalent. By Proposition 7,b, the isomorphism $*: S^2(V) \to S^{m-2}(V)$ is $\mathfrak{so}(V)$-equivariant. Clearly, it is also $\mathrm{SO}(V)$-equivariant and the action of $\mathrm{SO}(V)$ on $S^2(V)$ is equivalent to the adjoint action of $\mathrm{SO}(V)$. (Note that $S^2(V) \simeq \mathfrak{so}(V)$ by Proposition 6.) The result follows from Lemma 2. $\square$

Example 2. Assume that $m \geq 4$. Any m-dimensional invariant real or complex $(m-2)$-ary algebra has the form (V, μ), where $\mu = *(v)$ and $v \in V$. This follows from the existence of the isomorphism $*: V \to \bigwedge^{m-1} V$.

Assume that $\mathbb{K} = \mathbb{R}$. It is well-known that any real skew-symmetric matrix A can be written in the form $A = QA'Q^{-1}$, where

$$A' = \mathrm{diag}(J_{a_1}, \ldots, J_{a_k}, 0), \quad m = 2k+1,$$

and

$$A' = \mathrm{diag}(J_{a_1}, \ldots, J_{a_k}), \quad m = 2k.$$

Here

$$J_{a_j} = \begin{pmatrix} 0 & a_j \\ -a_j & 0 \end{pmatrix}, \quad a_j \in \mathbb{R},$$

and $Q \in \mathrm{SO}(V)$. Moreover we can assume that $0 \le a_k \le \cdots \le a_1$ if m is odd and $|a_k| \le \cdots \le a_1$ if m is even. Therefore, coadjoint orbits are parametrized by the numbers (a_j).

Let (ξ_i) be an orthogonal basis of V such that the matrix $A \in S^2(V) \simeq \mathfrak{so}(V)$ has the form

$$A = \mathrm{diag}(J_{a_1}, \ldots, J_{a_k}, 0) \ \text{ or } \ A = \mathrm{diag}(J_{a_1}, \ldots, J_{a_k}).$$

Then the corresponding element in $S^2 V$ is

$$v_A = a_1 \xi_1 \xi_2 + \ldots, a_k \xi_{2k-1} \xi_{2k},$$

where a_j are as above. We obtained the following theorem:

Theorem 3. [**Classification of real invariant** $(m-3)$**-ary** m**-dimensional algebras**] *Real invariant* $(m-3)$*-ary* m*-dimensional algebras are parametrized by vectors*

$$v = a_1 \xi_1 \xi_2 + \ldots, a_k \xi_{2k-1} \xi_{2k},$$

where $0 \le a_k \le \cdots \le a_1$ *if* $m = 2k+1$ *and* $|a_k| \le \cdots \le a_1$ *if* $m = 2k$. *Explicitly such algebras are given by the derived potentials* μ_v, *where* $\mu_v = *(v)$.

5.2 Classification of real m-dimensional simple $(m-3)$-ary algebras with a positive definite invariant form

In this section we give a classification of real m-dimensional simple $(m-3)$-ary algebras with an invariant form up to orthogonal isomorphism. Let V be a pure odd vector space over $\mathbb{R}$ with an even non-degenerate skew-symmetric form $(\,,)$. That is $V = V_{\bar{1}}$ and $(a,b) = (b,a)$ for all $a, b \in V$. We assume in addition that $(\,,)$ is positive definite.

Definition 10. A vector subspace $W \subset V$ is called an *ideal* of a symmetric n-ary algebra (V, μ) if $\mu(V, \ldots, V, W) \subset W$.

In other words, the vector space W is an ideal if and only if it is invariant with respect to the set of endomorphisms $\mu(v_1, \ldots, v_{n-1}, -) : V \to V$, where $v_i \in V$. Clearly, the vector space W is an ideal if and only if it is invariant with respect to the Lie algebra $\mathfrak{g}$ generated by all $\mu(v_1, \ldots, v_{n-1}, -)$.

Definition 11. A symmetric n-ary algebra (V, μ) is called *simple* if it is not 1-dimensional and if it does not have any proper ideals.

Example 3. The classification of simple complex and real Filippov n-algebras was done in [Ling]: there is one series of complex Filippov n-algebras A_k, where k is a natural number and several real forms for each A_k. The complex n-ary

algebra A_k has an invariant form and in our terminology it is given by the derived potential L (a top form) and formula (7).

Example 4. Let $m = 5$. By Theorem 3 we see that we have three types of 2-ary algebras up to isomorphism (μ_i is a derived potential):

- $\mu_1 = *(0) = 0$;

- $\mu_2 = *(a_1\xi_1\xi_2) = b_1\xi_3\xi_4\xi_5$, where $b_1 = \pm a_1 \neq 0$;

- $\mu_3 = *(a_1\xi_1\xi_2 + a_2\xi_3\xi_4) = b_1\xi_3\xi_4\xi_5 + b_2\xi_1\xi_2\xi_5$, where $b_1 = \pm a_1 \neq 0$ and $b_2 = \pm a_2 \neq 0$.

Obviously, the zero algebra $(V, 0)$ is not simple. The derived potential $\mu_2 = b_1\xi_3\xi_4\xi_5$ corresponds to the algebra with a non-trivial center:

$$(\mu_2)_{\xi_1} = (\mu_2)_{\xi_2} = 0.$$

Therefore, $[x, [\xi_i, \mu]] = 0$, $i = 1, 2$, for any $x \in V$. Hence $\langle \xi_1, \xi_2 \rangle$ is an ideal and the algebra (V, μ_2) is not simple. We will see that the algebra μ_3 is simple. It is not a Lie algebra because $[\mu_3, \mu_3] = -2b_1b_2\xi_1\xi_2\xi_3\xi_4 \neq 0$.

Theorem 4. [Classification of real simple $(m-3)$-ary algebras with an invariant form] *Assume that $m > 4$. All real m-dimensional $(m-3)$-ary algebras from Theorem 3 are simple except of two cases:*

- $v = 0$;

- $v = a_1\xi_1\xi_2$, *where* $a_1 \neq 0$.

Proof. As in the 2-ary case we can show that if W is an ideal in V, then $W^\perp$ is also an ideal in V. Let us take any real m-dimensional $(m-3)$-ary algebra (V, μ) and let us assume that W is an ideal. Then $V = W \oplus W^\perp$ and we have the decomposition:

$$\mu = \sum_t \mu_t, \quad \text{where} \quad \mu_t \in \overset{t}{\bigwedge} W \wedge \overset{m-2-t}{\bigwedge} W^\perp.$$

Since W and $W^\perp$ are ideals, we have $\mu_s = 0$ for $s \neq 0, m-2$. Assume that $\mu \neq 0$, then one of these ideals has dimension greater than or equal to $m-2$. Hence, we can assume that $\dim W = 1$ or 2. In case $\dim W = 1$, we have $\mu_{m-2} = 0$ and $(W^\perp, \mu_0)$ is an $(m-1)$-dimensional $(m-3)$-ary algebra, where $\mu_0 = *(w)$ and $w \in W^\perp$, see Example 2. The algebra $(W^\perp, \mu_0)$ has a zero ideal $\langle w \rangle$, since $[w, \mu_0] = 0$. Therefore, we can assume that $\dim W = 2$. In case $\dim W = 2$,

we have $\mu_{m-2} = 0$ and $(W^\perp, \mu_0)$ is an $(m-2)$-dimensional $(m-3)$-ary algebra. Hence, $\mu_0 \in \bigwedge^{top} W^\perp$. If $\mu_0 \neq 0$, then the algebra $(W^\perp, \mu_0)$ is simple, see Example 3. Therefore, any real invariant $(m-3)$-ary algebra with a proper ideal has the form $(V, 0)$ or $(V, *(a_1\xi_1\xi_2))$. All other algebras are simple.$\square$

5.3 Classification of real m-dimensional invariant simple $(m-3)$-ary algebras satisfying Jacobi identity 1 and 2

In this Section we classify real simple n-ary algebras with a positive definite invariant form satisfying Jacobi identity 1 and 2.

Jacobi identity 1. In [Ling] it was proven that there exist only one complex Filippov n-algebra for any $n > 2$. This algebra is $(n+1)$-dimensional. In our notations it is given by $*(1) = L$. Another result in [Ling] is the following:

A real simple Filippov n-algebra is isomorphic to the realification of a simple complex Filippov n-algebra or to a real form of a simple complex Filippov n-algebra.

In particular real simple Filippov n-algebras are of dimension $n+1$ or $2n+2$. It follows that simple n-ary algebras in Theorem 4 are not of Filippov type. For $n = m-2$ any derived potential has the form $\mu = *(v)$, where $v \in V \setminus \{0\}$. All such algebras have non-trivial centers because $[v, \mu] = 0$. Therefore, they are not simple. Furthermore, such algebras are of Filippov type. Indeed, since L satisfy the Jacobi identity (1) by Proposition 3, we have $[L_{a_1,\ldots,a_{m-1}}, L] = 0$ for any $a_i \in V$. Furthermore, for $\mu = *(v) = [v, L]$ we get

$$[\mu_{a_1,\ldots,a_{m-2}}, \mu] = [L_{a_1,\ldots,a_{m-2},v}, L_v] = [v, [L_{a_1,\ldots,a_{m-2},v}, L]] = 0.$$

By Proposition 3, we see that (V, μ) is a Filippov algebra. By the same argument the derived potential $[v, [w, L]]$ also corresponds to a Filippov algebra.

Theorem 5. *Assume that $m > 4$. Real m-dimensional Filippov n-algebras with a positive definite invariant form, where $n = m-1$, $m-2$ or $m-3$, are given up to $\mathrm{SO}(V)$-isometry by the following derived potentials:*

- $\mu = *(0) = 0$;

- $\mu = *(a \cdot 1) = a\xi_1 \cdots \xi_m$, *where $a \in \mathbb{R} \setminus \{0\}$;*

- $\mu = *(a\xi_1) = a\xi_2 \cdots \xi_m$, *where $a \in \mathbb{R}^{>0}$;*

- $\mu = *(a\xi_1\xi_2) = -a\xi_3 \cdots \xi_m$, *where $a \in \mathbb{R}^{>0}$.*

Jacobi identity (2). As above assume that $m > 4$ and $(,)$ is a symmetric positive definite form.

Theorem 6. *All algebras in Theorem 3 satisfy Jacobi identity (2) with the exception of the following cases:*

- $m = 5$, *the algebras with the derived potential* $\mu = *(a_1\xi_1\xi_2 + a_2\xi_3\xi_4)$, *where* $a_1, a_2 \neq 0$;

- $m = 6$, *the algebras with the derived potentials* $\mu = *(a_1\xi_1\xi_2 + a_2\xi_3\xi_4)$ *and* $\mu = *(a_1\xi_1\xi_2 + a_2\xi_3\xi_4 + a_3\xi_5\xi_6)$, *where* $a_i \neq 0$;

Proof. Assume that m is odd. By Corollary of Proposition 2 in this case Jacobi identity 2 is equivalent to $[\mu, \mu] = 0$. Assume that $m > 5$, then $[\mu, \mu] \in S^{2m-6}V = \{0\}$. In the case $m = 5$ the result follows from Example 4.

Assume that m is even. First of all consider the case $m = 6$. Let us take

$$\mu = b_1\xi_3\xi_4\xi_5\xi_6 + b_2\xi_1\xi_2\xi_5\xi_6, \quad b_1, b_2 \neq 0.$$

Denote by LHS the left hand side of (2). Let us calculate LHS for $(\xi_1, \ldots, \xi_5)$.

$$LHS = \{\{\xi_1, \xi_2, \xi_5\}, \xi_3, \xi_4\} + \{\{\xi_3, \xi_4, \xi_5\}, \xi_1, \xi_2\} = -2b_1b_2\xi_5 \neq 0.$$

The main idea here is to use the fact that $\{x, y, z\} = 0$ if $x \in \{\xi_1, \xi_2\}$ and $y \in \{\xi_3, \xi_4\}$. The proof for

$$\mu = b_1\xi_3\xi_4\xi_5\xi_6 + b_2\xi_1\xi_2\xi_5\xi_6 + b_3\xi_1\xi_2\xi_3\xi_4, \quad b_i \neq 0$$

is similar.

Consider the case m is even and $m > 6$. Let us compute LHS for (v_i), where $i = 1, \ldots, 2m-7$. Without loss of generality we can assume that between elements v_i are at least two equal. Let $v_s = v_t = v$. Clearly, $\{v_{i_1}, \ldots, v, \ldots, v, \ldots, v_{i_n}\} = 0$. Therefore,

$$LHS = \sum_{k,l} J_1^{(k,l)} + \sum_{k,l} J_2^{(k,l)},$$

where $J_1^{(k,l)}$ and $J_2^{(k,l)}$ is the sum of all summands of the form

$$\{\{v_{i_1}, \ldots, \underset{k}{v_s}, \ldots, v_{i_{m-3}}\}, v_{j_1}, \ldots, \underset{l}{v_t}, \ldots, v_{j_{m-4}}\},$$
$$\{\{v_{i_1}, \ldots, \underset{k}{v_t}, \ldots, v_{i_{m-3}}\}, v_{j_1}, \ldots, \underset{l}{v_s}, \ldots, v_{j_{m-4}}\}$$

respectively. Further,

$$J_1^{(k,l)} = \pm\sum(-1)^{(I,J)}\{\{v_{i_1}, \ldots, \underset{k}{\hat{v}_s}, \ldots, v_{i_{m-3}}, v_s\}, v_{j_1}, \ldots, \underset{l}{\hat{v}_t}, \ldots, v_{j_{m-4}}, v_t\} =$$
$$\pm\sum(-1)^{(I',J')}\{\{v_{i_1}, \ldots, \underset{k}{\hat{v}_s}, \ldots, v_{i_{m-3}}\}v, v_{j_1}, \ldots, \underset{l}{\hat{v}_t}, \ldots, v_{j_{m-4}}\}v,$$

where $\{\ldots\}_v$ is the multiplication corresponding to the derived potential $\mu_v = [v, \mu]$ and $(-1)^{(I', J')}$ is the sign of the permutation

$$(v_1, \ldots, \hat{v}_s, \ldots, \hat{v}_t, \ldots, v_{2m-7}) \longmapsto (v_{i_1}, \ldots, \underset{k}{\hat{v}_s}, \ldots, v_{i_{m-3}}, v_{j_1}, \ldots, \underset{l}{\hat{v}_t}, \ldots, v_{j_{m-4}}).$$

Since $\mu_v \in S^{m-3}W$, where $W = \langle v \rangle^\perp$, we see that $[\mu_v, \mu_v] = 0$. Therefore (2) holds for $\{\ldots\}_v$ and $J_1^{(k,l)} = 0$. Similarly, $J_2^{(k,l)} = 0$. The proof is complete.$\square$

Corollary. *All simple algebras from Theorem 4 satisfy Jacobi identity (2) for $m > 6$.*

6 Quasi-Frobenius skew-symmetric n-ary algebras

Let V be a pure odd vector space and $\mu \in S^n(V^*) \otimes V$ be an n-ary symmetric algebra structure on V.

Definition 12. An n-ary algebra (V, μ) is called *quasi-Frobenius* if it is equipped with a symmetric bilinear form φ such that

$$\sum_{cycl} \varphi(a_1, \mu(a_2, \ldots, a_{n+1})) = 0. \tag{12}$$

If we forget about superlanguage this means that the algebra (V, μ) is skew-symmetric and φ is a skew-symmetric bilinear form on V.

Example 5. Assume that $n = 2$ and (V, μ) is a Lie algebra. Then our definition coincides with the definition of a quasi-Frobenius Lie algebra. Recall that a *quasi-Frobenius Lie algebra* is a Lie algebra $\mathfrak{g}$ equipped with a non-degenerate skew-symmetric bilinear form β such that

$$\beta([x, y], z) + \beta([z, x], y) + \beta([y, z], x) = 0.$$

We may assign an n-ary algebra $(V \oplus V^*, \mu^T)$ to (V, μ), called the T_0^*-extension of (V, μ). (The notion of T_θ^*-extension for algebras was introduced and studied in [Bord]. We will need this notion only for $\theta = 0$.) The construction of $(V \oplus V^*, \mu^T)$ is the following: the n-ary algebra structure μ^T is just the image of μ by the natural inclusion $S^n(V^*) \otimes V \hookrightarrow S^*(V^* \oplus V)$. Furthermore, the pure odd vector space $V \oplus V^*$ has a skew-symmetric (in supersense) pairing given by

$$(a, \alpha) = (\alpha, a) = \alpha(a),$$

where $\alpha \in V^*$ and $a \in V$. This defines a Poisson bracket on $S^*(V \oplus V^*)$. So $(V \oplus V^*, \mu^T)$ as a quadratic symmetric n-ary algebra, where the multiplication is given by the derived bracket with the derived potential $\mu^T \in S^*(V^* \oplus V)$. More precisely, the new multiplication μ^T in $V \oplus V^*$ is given by:

$$\mu^T|_{S^n(V)} = \mu, \quad \mu^T|_{S^{n-k}(V) \cdot S^k(V^*)} = 0 \text{ if } k > 1, \quad \mu^T(S^{n-1}(V) \cdot S^1(V^*)) \subset V^*$$

and

$$\mu^T(a_1, \ldots, a_{n-1}, b^*)(c) := -b^*(\mu(a_1, \ldots, a_{n-1}, c)).$$

The main observation here is:

Proposition 11. *Let V be a pure odd vector space and n be even. Then an n-ary algebra (V, μ) has a quasi-Frobenius structure with respect to a symmetric form φ if and only if the maximal isotropic subspace $B_\varphi = \{a + \varphi(a, -)\} \subset V \oplus V^*$ is a subalgebra in $(V \oplus V^*, \mu^T)$. In other words, there is a one-to-one correspondence between quasi-Frobenius structures on (V, μ) and maximal isotropic subalgebras in $(V \oplus V^*, \mu^T)$ that are transversal to V^*.*

Proof. First of all it is well-known that maximal isotropic subspaces in $V \oplus V^*$ that are transversal to V^* are in one-to-one correspondence with $\varphi \in S^2V$. Let us show that φ satisfies (12) if and only if B_φ is a subalgebra. Denote $a^* := \varphi(a, -) \in V^*$. Then we have:

$$(\mu^T(a_1 + a_1^*, \ldots, a_n + a_n^*), c + c^*) =$$
$$c^*(\mu(a_1, \ldots, a_n)) + \sum_k (\mu^T(a_1, \ldots, a_k^*, \ldots, a_n), c) =$$
$$\varphi(c, \mu(a_1, \ldots, a_n)) - \sum_k a_k^*(\mu(a_1, \ldots, a_{k-1}, c, a_{k+1}, \ldots, a_n)) =$$
$$\varphi(c, \mu(a_1, \ldots, a_n)) - \sum_k \varphi(a_k, \mu(a_1, \ldots, a_{k-1}, c, a_{k+1} \ldots, a_n)).$$

Furthermore,

$$\varphi(a_k, \mu(a_{k+1}, \ldots, a_n, c, a_1 \ldots, a_{k-1})) =$$
$$(-1)^{(k-1)(n-k-1)} \varphi(a_k, \mu(a_1, \ldots, a_{k-1}, a_{k+1} \ldots, a_n, c)) =$$
$$(-1)^{k(n-k-1)+1} \varphi(a_k, \mu(a_1, \ldots, a_{k-1}, c, a_{k+1} \ldots, a_n)).$$

If n is even, $(-1)^{k(n-k-1)+1} = -1$. Therefore, we have:

$$(\mu^T(a_1 + a_1^*, \ldots, a_n + a_n^*), a_{n+1} + a_{n+1}^*) = \sum_{cycl} \varphi(a_1, \mu(a_2, \ldots, a_{n+1})).$$

This expression is equal to 0 if and only if the algebra (V, μ) is quasi-Frobenius with respect to φ. On other side, $(\mu^T(a_1 + a_1^*, \ldots, a_n + a_n^*), a_{n+1} + a_{n+1}^*)$ is equal to 0 if and only if B_φ is a subalgebra in $(V \oplus V^*, \mu^T)$. The proof is complete.$\square$

Remark. The result of Proposition 11 is well-known for Lie algebras.

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